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Little-Group Approach to Gauge Theory

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Factor representations of the nonsemisimple $E(2)$-like little group for massless particles are used to construct free gauge theories for arbitrary spin, which explicitly solves the dichotomy between the unitary requirement and the appearance of gauge degrees of freedom. This conceptually new approach may help us understand the deep relation between gauge theory and space-time symmetry.

In a fundamental paper Wigner[(1)] gave a complete classification of unitary irreducible representations of the Poincaré group based on the little-group method. The relation between his results and the construction of relativistic wave equations has already been discussed.[(2–5)] However, it is still unclear how gauge theories for massless particles could be fitted into this picture. For example, in ref. 5, it is claimed that all the covariant wave functions corresponding to a given mass and spin are characterized by a finite-dimensional representation of the Lorentz group and a spin projection, and that, in the massless case, the spin projection is uniquely fixed by the unitary condition.[2] A relativistic wave equation is just the covariant expression of the spin projection in the coordinate representation. But the vector-potential description of the photon certainly lies outside this kind of construction since the fixed spin projection allows for no gauge degrees of freedom. On the other hand, it has been suggested that the appearance of gauge degrees of freedom should originate from the nonsemisimple structure of the $E(2)$-like little group for massless particles and that the translational degrees of freedom in $E(2)$ should correspond to gauge transformations.[(6–9)] Since a conceptually

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[2] Weinberg[(4)] proved a similar theorem, claiming that only the spin $j = |A - B|$ massless field can be constructed from a representation (A, B) of the Lorentz group.

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clear and complete analysis of this problem is lacking, it is the purpose of the present paper to try to fill the gap.

First let us recall that, by the little-group method we mean a way to construct unitary irreducible representations of the Poincaré group from the unitary irreducible representations of the little group, which leaves invariant the four-momentum of a massive or massless particle, with orbit completion.[10]

If we choose the four-momentum of a given free massless particle along the z direction, i.e.,

$$p_1 = p_2 = 0, \qquad p_3 = -p_0 \tag{1}$$

then the generators of the little group are[8]

$$N_1 = K_1 - J_2, \qquad N_2 = K_2 + J_1, \qquad J_3 \tag{2}$$

where $\bar{J}$ and $\bar{K}$ are the generators of the rotations and Lorentz boosts, respectively. It is easy to verify that

$$[J_3, N_1] = iN_2 \tag{3}$$

$$[J_3, N_2] = -iN_1 \tag{4}$$

$$[N_1, N_2] = 0 \tag{5}$$

so that the Lie algebra of the little group is isomorphic to that of the 2-dimensional Euclidean group $E(2)$. Since the two "translations" N_1 and N_2 form an invariant abelian subalgebra, the $E(2)$-like little group is not semisimple.

Now the matrix representation of the generators on the four-vector space takes the form[8]

$$N_1 = \begin{bmatrix} 0 & 0 & -i & i \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix} \tag{6}$$

$$N_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & i \\ 0 & i & 0 & 0 \\ 0 & i & 0 & 0 \end{bmatrix} \tag{7}$$

$$J_3 = \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{8}$$

If we want to construct a four-potential theory for the photon, we should start from this representation of the little group, which is obviously reducible. However, due to the nonsemisimple structure of $E(2)$, the representation is not fully reducible.[9] Let us choose a convenient basis for the four-vector space V as follows:

$$\epsilon_+ = \begin{bmatrix} 1 \\ i \\ 0 \\ 0 \end{bmatrix}, \qquad \epsilon_- = \begin{bmatrix} 1 \\ -i \\ 0 \\ 0 \end{bmatrix}, \qquad n = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \qquad \hat{p} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \tag{9}$$

Then we have

$$N_1 n = i(\epsilon_+ + \epsilon_-), \qquad N_2 n = \epsilon_+ - \epsilon_-, \qquad J_3 n = 0 \tag{10}$$

$$N_1 \epsilon_+ = i\hat{p}, \qquad N_2 \epsilon_+ = -\hat{p}, \qquad J_3 \epsilon_+ = \epsilon_+ \tag{11}$$

$$N_1 \epsilon_- = i\hat{p}, \qquad N_2 \epsilon_- = \hat{p}, \qquad J_3 \epsilon_- = -\epsilon_- \tag{12}$$

$$N_1 \hat{p} = 0, \qquad N_2 \hat{p} = 0, \qquad J_3 \hat{p} = 0 \tag{13}$$

Let us denote S_0, S_+, S_-, S as the subspaces spanned by $\{\hat{p}\}$, $\{\epsilon_+, \hat{p}\}$, $\{\epsilon_-, \hat{p}\}$, $\{\epsilon_+, \epsilon_-, \hat{p}\}$, respectively. Then we immediately find the following lattice of invariant subspaces:

$$\begin{array}{ccccc} & \nearrow & S_+ & \searrow & \\ V \rightarrow S & & & & S_0 \rightarrow \{0\} \\ & \searrow & S_- & \nearrow & \end{array} \tag{14}$$

where the arrows represent inclusion. The only irreducible subspace is S_0, on which the little group acts trivially. The corresponding scalar theory is the only one that can be obtained by the spin projection method of ref. 5. It is here that we introduce a conceptually new construction which enables us to incorporate the vector potential gauge theory of the photon. The idea is to use the factor representation[3] of the little group. This is a standard procedure to obtain the irreducible components of a reducible, yet not fully reducible (decomposable) representation. For example, if the matrix representation T of a group $\mathscr{G}$ is of the form

[3] See, for example, Kirillov.[11] However, this concept is not discussed in most of the group theory textbooks for physicists.

$$T(a) = \begin{bmatrix} T_1(a) & 0 \\ Q(a) & T_2(a) \end{bmatrix}, \qquad a \in \mathcal{G} \tag{15}$$

and the subrepresentation T_2 is carried by the subspace W_2 of the whole space W, then the subrepresentation T_1 can only be carried by the factor space $W_1 = W/W_2$. In our case, we can build the plus (minus) helicity representation of the $E(2)$-like little group on the factor space S_+/S_0 (S_-/S_0). For photons, parity is conserved, so we should include the two helicity states in one representation on S/S_0. It is obvious that what we have factored out are just vectors proportional to the four-momentum p since $\hat{p}$ is proportional to p, and this corresponds to factoring out gauge transformations in the momentum representation. Note that this kind of gauge transformation is produced by the "translations" in the $E(2)$-like little group [see Eqs. (11) and (12)]. Therefore the use of factor representation has enabled us to get a physically needed unitary representation of the little group with the "translations" represented by unity and thus solves the dichotomy between gauge degrees of freedom and the unitary requirement, which is the main confusing point in previous discussions.[4] For a covariant description, we notice that $S = \{A^\mu \in V \big| p_\mu A^\mu = 0\}$ and $S_0 = \{\lambda p^\mu \big| \lambda \in R\}$. The representation of the little group on the factor space S/S_0 is isomorphic to that on the tensor space $\{F^{\mu\nu} = p^\mu A^\nu - p^\nu A^\mu \big| A^\mu \in S\}$,[(12),5] which has no gauge degree of freedom in it. However, when considering interactions, the vector-potential description should be the right choice.[(6,14)]

This construction can be generalized to arbitrary spin. Notice that what we should do is to construct unitary irreducible representations of the little group with proper helicities, restricted from irreducible representations of the Lorentz group. Let us recall that the rotation and boost generators $\vec{J}$, $\vec{K}$ satisfy the following commutation relations[(4)]:

$$[J_i, J_j] = i\varepsilon_{ijk} J_k \tag{16}$$

$$[J_i, K_j] = i\varepsilon_{ijk} K_k \tag{17}$$

$$[K_i, K_j] = -i\varepsilon_{ijk} K_k \tag{18}$$

If we define $A_i = \frac{1}{2}(J_i + iK_i)$, $B_i = \frac{1}{2}(J_i - iK_i)$, then we have

$$[A_i, A_j] = i\varepsilon_{ijk} A_k \tag{19}$$

$$[B_i, B_j] = i\varepsilon_{ijk} B_k \tag{20}$$

[4] For example, the unitary requirement is abandoned in ref. 5 to incorporate the vector gauge theory.

[5] Ref. 12 uses the theory of vector bundles and induced representations, which should be the proper mathematical language for our discussion. For this aspect, see also Hermann.[(13)]

$$[A_i, B_j] = 0 \tag{21}$$

Therefore finite-dimensional irreducible representations of the Lorentz group can be labeled by (l, l') such that

$$A^2 = A_1^2 + A_2^2 + A_3^2 = l(l + 1) \tag{22}$$

$$B^2 = B_1^2 + B_2^2 + B_3^2 = l'(l' + 1) \tag{23}$$

We can choose a basis $\{\psi_{i,i'} \big| i = -l, \ldots, l; i' = -l', \ldots, l'\}$ such that

$$A_3\psi_{i,i'} = i\psi_{i,i'} \tag{24}$$

$$B_3\psi_{i,i'} = i'\psi_{i,i'} \tag{25}$$

From Eq. (2), we see that

$$N_1 = K_1 - J_2 = -i(A_- - B_+) \tag{26}$$

$$N_2 = K_2 + J_1 = A_- + B_+ \tag{27}$$

$$J_3 = A_3 + B_3 \tag{28}$$

Therefore we have

$$A_- = \tfrac{1}{2}(iN_1 + N_2) \tag{29}$$

$$B_+ = \tfrac{1}{2}(-iN_1 + N_2) \tag{30}$$

Knowing the action of N_1 and N_2 on the basis, we can now construct, similar to (14), a lattice of invariant subspaces $S_{i,i'}$ such that $S_{i,i'} \supset S_{j,j'}$ whenever $i \geq j$, $i' \leq j'$. The space $S_{i,i'}$ is just spanned by basis vectors $\psi_{j,j'}$ for $-l \leq j \leq i$, $i' \leq j' \leq l'$. While the only irreducible subspace is $S_{-ll'}$, there are as many as $(2l + 1)(2l' + 1)$ unitary irreducible components which may be built on factor spaces. For example, the helicity $(i + i')$ representation is built on the factor space $S_{i,i'}/(S_{i-1,i'} \oplus S_{i,i'+1})$. The translations N_1 and N_2 still generate "gauge transformations" which are properly factored out. If we introduce the exterior product $\wedge$, we can identify this representation with the one on the space $T_{i,i'}$ spanned by $\psi_{i,i'} \wedge \psi_{i-1,i'} \wedge \psi_{i,i'+1} \wedge \cdots \wedge \psi_{-l,l'}$.

To obtain a covariant description, we should start from tensors or tensor-spinors. For integer helicity $\pm j$, we can use symmetric tensors $A^{\mu_1 \cdots \mu_j}$ of order j, with zero trace

$$g_{\mu_1\mu_2} A^{\mu_1\mu_2\ldots\mu_j} = 0 \tag{31}$$

These tensors form an irreducible representation $(j/2, j/2)$ of the Lorentz group. After applying the projection condition

$$p_{\mu_1} A^{\mu_1 \cdots \mu_j} = 0 \tag{32}$$

we still have a big space containing tensors like

$$\begin{gathered} \epsilon_{\pm}^{\mu_1} \ldots \epsilon_{\pm}^{\mu_j} \\ \{\epsilon_{\pm}^{\mu_1} \ldots p^{\mu_k} \ldots \epsilon_{\pm}^{\mu_j}\} \\ \vdots \\ p^{\mu_1} \ldots p^{\mu_j} \end{gathered} \tag{33}$$

where { } represents symmetrization. Therefore we should factor out all the tensors containing the four-momentum p^{μ}, which can be generated from $\epsilon_{\pm}^{\mu_1} \ldots \epsilon_{\pm}^{\mu_j}$ by the action of the translations N_1 and N_2. This kind of construction of gauge theories has already been discussed and used to derive the electromagnetic and gravitational field equations.(14,15)

For half-integer helicity $j + \frac{1}{2}$, we can start from a traceless symmetric tensor-spinor $\psi^{\mu_1 \cdots \mu_j}$ satisfying

$$g_{\mu_1 \mu_2} \psi^{\mu_1 \cdots \mu_j} = 0 \tag{34}$$

Now the spin projection conditions are

$$\gamma_{\mu_1} \psi^{\mu_1 \cdots \mu_j} = 0 \tag{35}$$

$$\gamma_{\mu} p^{\mu} \psi^{\mu_1 \cdots \mu_j} = 0 \tag{36}$$

$$p_{\mu_1} \psi^{\mu_1 \cdots \mu_j} = 0 \tag{37}$$

Notice that Eq. (37) can be derived from Eqs. (35) and (36). The restricted subspace is spanned by

$$\begin{gathered} \chi_{\pm} \epsilon_{\pm}^{\mu_1} \ldots \epsilon_{\pm}^{\mu_j} \\ \chi_{\pm} \{\epsilon_{\pm}^{\mu_1} \ldots p^{\mu_k} \ldots \epsilon_{\pm}^{\mu_j}\} \\ \vdots \\ \chi_{\pm} p^{\mu_1} \ldots p^{\mu_j} \end{gathered} \tag{38}$$

where $\chi_{\pm}$ are Dirac spinors with helicities $\pm\frac{1}{2}$, satisfying $\gamma_{\mu} p^{\mu} \chi_{\pm} = 0$. Again we should factor out all the terms containing p^{μ}. Similar discussions can be found in the literature.(16)

From the above discussions, we see the importance of the four-vector representation as a building block. However, since the left and right spinor representations are more fundamental, we should discuss them. The $E(2)$-like little group is now generated by

$$T_1 = \mp\tfrac{i}{2}\sigma_1 - \tfrac{1}{2}\sigma_2 \tag{39}$$

$$T_2 = \tfrac{1}{2}\sigma_1 \mp\tfrac{i}{2}\sigma_2 = -(\mp)iT_1 \tag{40}$$

$$S_3 = \tfrac{1}{2}\sigma_3 \tag{41}$$

where $-$ $(+)$ corresponds to the left (right) spinor. Let us choose a basis

$$\psi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \psi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{42}$$

Then we have

$$S_3\psi_+ = \tfrac{1}{2}\psi_+, \qquad S_3\psi_- = -\tfrac{1}{2}\psi_- \tag{43}$$

$$T_1\psi_- = 0, \qquad T_1\psi_+ = -i\psi_- \qquad \text{(for left spinor)} \tag{44}$$

$$T_1\psi_+ = 0, \qquad T_1\psi_- = i\psi_+ \qquad \text{(for right spinor)} \tag{45}$$

Therefore, there is no gauge problem for the left spinor ψ_- or the right spinor ψ_+, and they can be used to construct the well-known "two-component theory of neutrinos."[(8)] But the left spinor ψ_+ and the right spinor ψ_- correspond to gauge spinors which have seldom been discussed (see, however, ref. 17).

In conclusion, we have shown that gauge theories are connected with the factor representations of the $E(2)$-like little group for massless particles. The spin projection scheme described in ref. 5 should be supplemented by factoring out a suitable invariant subspace when discussing massless particles. Gauge transformations are closely related to the "translations" in the $E(2)$-like little group. These results may be helpful for a deeper understanding of the gauge principle and its relation to space-time symmetry.

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